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Brief Communication

On the separation of a suspension in a tube centrifuge

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1. Introduction

Consider the batch separation of a suspension in a cylindrical tube which rotates about an axis perpendicular to the axis of the tube. This problem is relevant to devices used in biology and medicine laboratories. The full solution is bound to be very complicated because of the many effects involved, but useful approximations can be readily derived from the available theory (Ungarish, 1993). Anestis and Schneider (1983) made such an attempt, but they used a simplification which led to physically unacceptable (or at least hard for interpretation) results. Stibi and Schaflinger (1991) also made an indirect contribution to this problem, but the pertinent results are obscured by the different scope of their study.

The objective of this note is to clarify briefly the essentials of the analysis of this process and to present several simple results that may be useful to engineers and to further investigations.

2. Analysis

Consider the straight tube centrifuge sketched in Fig. 1. The Cartesian xyz system, co-rotating with the tube with constant angular velocity Ω about the axis z, is used. The coordinate x is taken along the axis (centerline) of the tube. The cross-section area of the tube, in the yz plane, is constant; let us denote by A the area, by C the bounding curve and by D the typical width (for example, $A^{1/2}$) of this geometry. The length of the container is L. The boundaries $x = x_i$ and $x = R = x_i + L$ of the considered container are referred to as the inner and outer walls, respectively, and the tube provides the side walls of the centrifuge. Usually, the tube container is slender, $L \gg D$, and located not far from the axis of rotation, $x_i/L < 1$, but these assumptions are not

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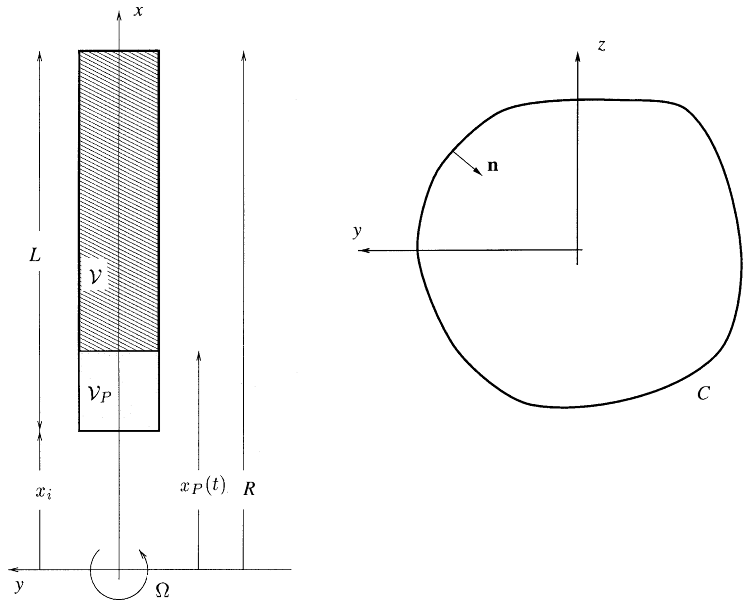

Fig. 1. Schematic view of a tube centrifuge rotating about the axis z. The shock $x_P(t)$ divides between the pure fluid domain (white) and the suspension of "heavy" particles domain (gray). The thin sediment layers are not shown. The cross-section yz (enlarged) has a constant area A and bounding curve C; the normal vector $\hat{\mathbf{n}}$ is also shown. The boundaries $x = R$ and $x = x_i$ are called "outer" and "inner" walls, and the tube provides the "side walls".

essential to the analysis. We assume $x_i \gg D$ for achieving some simplifications of the discussion, in particular concerning curvature terms.

The container is filled with a monodispersed suspension of "heavy" small particles (initial volume fraction $\alpha(0)$) and, at $t = 0$, is set rapidly into rotation with constant Ω. We assume that solid body rotation is established quickly, before any significant separation occurs, and that the influence of the gravity acceleration g can be neglected (the criterion is discussed later). The suspension in the tube is therefore subjected to the body force introduced by the centrifugal acceleration $\Omega^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$; this is counteracted by a pressure distribution which keeps the fluid in solid body rotation with the tube (within a small deviation not considered here).

On the other hand, the heavy dispersed particles are not sufficiently restrained by that pressure field and start to move in the direction of the body force. The excess centrifugal drive (a buoyant force) on the particle is balanced by the Stokes drag induced by the velocity of the particle relative to the embedding fluid. This buoyancy–drag balance indicates that the "dispersed phase" moves with respect to the "continuous phase" with the relative velocity

$$\mathbf{v}_R = Ub(\alpha)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}), \tag{1}$$

where

$$U = \epsilon\beta\Omega \quad \text{and} \quad b(\alpha) = (1 - \alpha)/\mu(\alpha), \tag{2}$$

$$\epsilon = (\rho_D - \rho_C)/\rho_C, \quad \beta = (2/9)a^2\Omega/\nu; \tag{3}$$

here ρ is the density, the subscripts D and C denote the dispersed and continuous phases, a is the radius of the particle, v the kinematic viscosity of the clean suspending fluid, α is the volume fraction of particles, and $\mu(\alpha)$ is the effective viscosity ratio correlation, usually taken as $(1-\alpha)^{-m}$, with $m = 2.5$ or 3.1.

It is important to notice that $\mathbf{v}_R$ has a component in the lateral direction $\hat{\mathbf{y}}$ which is an odd function of y. Although this component is relatively small as compared with the component in the longitudinal direction $\hat{\mathbf{x}}$ in a slender tube ($D/L \ll 1$), it's contribution to the separation process turns out to be significant.

The separation features are as follows. The particles are driven out from the bulk of suspension according to the above relative velocity, away from the inner wall (i) and towards the side walls (sw) and outer wall (ow). A region of pure fluid $x_\mathrm{i} \leqslant x \leqslant x_\mathrm{P}(t)$ and layers of sediment (assumed very thin) appear.

The local volume fraction in the domain of suspension decays. This feature is governed by the particle conservation equation (see Ungarish, 1993, Section 2.5)

$$\frac{\partial \alpha}{\partial t} + [\mathbf{j} + (1-2\alpha)\mathbf{v}_R] \cdot \nabla \alpha = -\alpha(1-\alpha)\nabla \cdot \mathbf{v}_R, \quad (4)$$

subject to the initial conditions $\alpha = \alpha(0)$ in the domain of interest, where $\mathbf{j}$ is the volume velocity of the mixture (recall: $\mathbf{j} = \alpha \mathbf{v}_\mathrm{D} + (1-\alpha)\mathbf{v}_\mathrm{C}$, $\mathbf{v}_R = \mathbf{v}_\mathrm{D} - \mathbf{v}_\mathrm{C}$ and $\nabla \cdot \mathbf{j} = 0$). In general, this equation can be solved by the method of characteristics. Substituting (1) in (4) and considering the initial conditions, we observe that all characteristics carry the same information and actually the solution α in the entire suspension domain is a function of t only and given by

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -2\epsilon\beta\Omega\Phi(\alpha), \quad \Phi(\alpha) = \alpha(1-\alpha)b(\alpha), \quad \text{with given } \alpha(0). \quad (5)$$

At the boundary of the suspension domain, jumps into pure fluid and sediment regions via stable kinematic shocks are assumed. The detailed solution is beyond the scope of this note, but the analysis is expected to be straightforward, at least for small concentrations. The numerical evaluation of $\alpha(t)$ from (5) is straightforward, but in the dilute limit $\alpha \to 0$, $\Phi(\alpha) = \alpha$, (5) yields the approximate result

$$\alpha(t) = \alpha(0)\exp(-2\epsilon\beta\Omega t). \quad (6)$$

The same results for the behavior of $\alpha(t)$ are obtained for the separation of a similar suspension in a long cylindrical centrifuge whose axis of symmetry coincides with the axis of rotation, z (Ungarish, 1993, Section 4.2).

During separation the (shrinking) volume occupied by the suspension is $\mathscr{V} = A[R - x_\mathrm{P}(t)]$ and the (expanding) volume of the pure fluid region is $\mathscr{V}_\mathrm{P} = A[x_\mathrm{P}(t) - x_\mathrm{i}]$ (again, the sediment layers are assumed to be very thin, and we neglect the small curvature of the shock, which is expected to be cylindrical around the axis z). Consider the rate of production of pure fluid in this centrifuge. As shown in Ungarish (1993), Section 2.3, in a closed container and when the volume fraction of the particles in the suspension domain is a function of t only, volume conservation of the components requires

$$\frac{\mathrm{d}\mathscr{V}_\mathrm{P}}{\mathrm{d}t} = \frac{1}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t}\mathscr{V} + J_\mathrm{s}, \quad (7)$$

where the first term of the RHS represents the contribution of the decay of α inside the suspension zone and the second term represents the contribution of the settling (or sedimentation) of particles on the boundaries. Since the volume of pure fluid cannot decrease and the first term of the RHS is negative, we conclude that settling on the boundaries must take place at least at the rate necessary for removing the particles expelled from the suspension due to the decay of α (the squeezing effect).

In the present configuration, the settling term has two contributions, from the side wall (denoted sw, the tube) and from the outer wall (denoted ow, the lid $x = R$), and can be expressed as

$$J_s = -(1-\alpha)\left[\int_{sw} \mathbf{v}_R \cdot \hat{\mathbf{n}}\, dA + \int_{ow} \mathbf{v}_R \cdot \hat{\mathbf{n}}\, dA\right], \tag{8}$$

where $\hat{\mathbf{n}}$ is the unit vector pointing into the container.

To evaluate the first integral in (8) we first notice that the normal vector to the side wall is in the yz plane, see Fig. 1, and hence only the $\hat{\mathbf{y}}$ component of $\mathbf{v}_R$, given by (1), contributes to the settling on the side wall. Also, we observe that settling takes place for $x \geqslant x_P(t)$ only. Therefore, we rewrite

$$I_{sw} = \int_{sw} \mathbf{v}_R \cdot \hat{\mathbf{n}}\, dA = \int_{x_P(t)}^{R} dx \left[\int_C Ub(\alpha) y \hat{\mathbf{y}} \cdot \hat{\mathbf{n}}\, ds\right], \tag{9}$$

where ds is the arc length differential along the bounding curve. Using Green's theorem, $-\oint_C \mathbf{V} \cdot \hat{\mathbf{n}}\, ds = \int\int \nabla \cdot \mathbf{V}\, dA$ (the $-$ sign is for the present inward direction of $\hat{\mathbf{n}}$) we obtain the compact result

$$I_{sw} = -Ub(\alpha)A[R - x_P(t)]. \tag{10}$$

It is remarkable that only the area A of the tube cross-section, but not the shape, is of importance in this result.

The evaluation of the second integral in (8) is straightforward. At the outer wall $x = R$ and $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, and using (1) we simply obtain

$$I_{ow} = \int_{ow} \mathbf{v}_R \cdot \hat{\mathbf{n}}\, dA = -Ub(\alpha)AR. \tag{11}$$

Substituting these results into (8) and using the definition of $\Phi(\alpha)$ given in (5) yields

$$J_s = \frac{1}{\alpha}\Phi(\alpha)UA[2R - x_P(t)]. \tag{12}$$

This expression is the contribution of the sedimentation to the rate of production of pure fluid. The noteworthy fact is that the settling on the side walls may be a very significant component of the process: the ratio I_{sw}/I_{ow} is L/R at the beginning of separation ($x_P = x_i = R - L$), but decays as $x_P(t)$ approaches R.

The result (12) can be expressed in the following convenient form, after the use of (5) to eliminate $\Phi(\alpha)$ and recalling $\mathscr{V} = A[R - x_P(t)]$, $U = \epsilon\beta\Omega$

$$J_s = -\frac{1}{\alpha}\frac{d\alpha}{dt}\left[\mathscr{V} + \frac{1}{2}Ax_P(t)\right]. \tag{13}$$

Next, substituting (13) into (7) yields

$$\frac{d\mathscr{V}_P}{dt} = -\frac{1}{2}\frac{1}{\alpha}\frac{d\alpha}{dt}Ax_P(t), \tag{14}$$

which, since $\mathscr{V}_P = A[x_P(t) - x_i]$, can be readily reduced to an equation for $x_P(t)$ and solved. The result, subject to the condition that the pure fluid shock starts its propagation at $x = x_i$, is

$$x_P(t) = x_i\left[\frac{\alpha(0)}{\alpha(t)}\right]^{1/2}, \tag{15}$$

with $\alpha(t)$ provided by the integration of (5). In the dilute limit, see (6),

$$x_P(t) = x_i \exp(\epsilon\beta\Omega t). \tag{16}$$

The separation is completed at the time t_{sc} when $x_P(t)$ reaches the position of the sediment layer on the outer wall. In the dilute case, the condition $x_P(t_{sc}) = R$ yields, after use of (16)

$$t_{sc} = -\frac{1}{\epsilon\beta\Omega}\ln\left(1 - \frac{L}{R}\right). \tag{17}$$

The representative separation time interval is $(\epsilon\beta\Omega)^{-1}$, during which a significant decay of the volume fraction α takes place.

Surprisingly, the same formulas for the position of the pure-fluid interface, $x_P(t)$, and separation time interval, t_{sc}, are obtained for a long axisymmetric cylindrical centrifuge of inner radius x_i and outer radius $R = x_i + L$ (again, the axis of rotation and symmetry is z, and x is regarded as the radius in a cylindrical system, see Ungarish, 1993, Section 4.2). In such a centrifuge, there are no side walls and obviously all the sedimentation is on the outer wall.

Suppose that the gravity acceleration acts in the $\hat{\mathbf{z}}$ direction. The contribution to the relative velocity is, approximately, $[(2/9)\epsilon a^2 g/\nu]\hat{\mathbf{z}}$. During the typical separation time interval, $(\epsilon\beta\Omega)^{-1}$, the corresponding displacement of a particle is g/Ω^2. If $g/\Omega^2 \ll D$ this effect can be ignored, otherwise gravitational sedimentation on the side wall must be taken into account.

3. Concluding remarks

The main conclusion is that for the separation process of a suspension of heavy particles, there is a remarkable similarity between the tube centrifuge and the cylindrical axisymmetric (or sector of it) centrifuge as regarding the behavior of $\alpha(t)$ and position of the inner pure fluid and outer sediment shocks. (The compared centrifuges rotate with the same Ω about the same axis, and have similar positions of the inner and outer boundaries.) On the other hand, the settling on the side walls of the tube has no counterpart in the axisymmetric cylindrical case. These features are straightforward consequences of the balance between the centrifugal buoyancy and the linear Stokes drag that is assumed to govern the motion of the dispersed particles. However, for very fast rotation and/or large particles (when β is not very small) the "postulate" (1) is invalid. These and other related effects require a different analysis along the lines indicated in Stibi and Schaflinger (1991) and Ungarish (1993).

Anestis and Schneider (1983) neglected the y component of the relative velocity, see (1), and obtained results which differ significantly from the present ones. Indeed, the y velocity component is small as compared with the x component when the tube is long and thin, $D/L \ll 1$, but nevertheless the contribution of this small effect (the sedimentation on the side walls) when integrated over the length of tube turns out to be important. Actually, a typical particle must travel only a small distance (say D) in the y direction in order to hit the side wall, but a much larger distance (say L) for reaching the outer wall, and this small distance ratio may compensate for the small velocity-component ratio, with the result that the aspect ratio parameter of the tube, D/L, does not affect the separation (assuming thin sediment layers, see below). Furthermore, if the y component is neglected in (1) the divergence of the centrifugal field is distorted and an error of 50% is introduced in the $\nabla \cdot \mathbf{v}_R$ term in the right-hand side of (4). In other words, in general, centrifugal separation cannot be treated as one-dimensional in Cartesian coordinates.

However, one would expect that in some limiting cases of the considered configuration, the centrifugal separation process in the tube is analogous to the more conventional gravity settling in a similar straight container (with the appropriate use of $\Omega^2 R$ instead of g). This analogy is indeed attained when the length of the tube is much smaller than the distance from the axis of rotation, $L \ll R$. In this case, the variation of the centrifugal driving force over the length of the container is relatively small, and the settling on the side wall is much smaller than on the outer wall (recall $I_{sw}/I_{ow} \leqslant L/R$). The expansion of (17) for small values of (L/R) yields, to leading order in this parameter,

$$t_{sc} \approx L/(\epsilon\beta\Omega R) \quad (L/R \ll 1), \tag{18}$$

and substitution of this result in (6) indicates that

$$\alpha(t_{sc}) \approx \alpha(0)(1 - 2L/R) \quad (L/R \ll 1). \tag{19}$$

Thus, in this limit the separation is essentially represented by the motion of the pure-fluid interface with constant velocity $(2/9)\epsilon a^2 \Omega^2 R/\nu$ in the x direction, while the volume fraction α in the suspension region is almost constant. But note that the same results are obtained for an axisymmetric cylindrical centrifuge of outer radius R and inner radius $R - L$ when $L/R \ll 1$, so it is not the tube geometry but rather the behavior of the centrifugal driving in a shallow layer of suspension (compared with the distance from the axis of rotation) that creates the analogy with the gravity separation.

The assumption that the sediment layers are very thin can be relaxed by replacing R with $x_S(t)$ and A with $A - \delta A(t, x)$ in the foregoing analysis, where x_S is the position of the sediment shock formed on the outer wall and δA is the area of the accumulated sediment on the side walls. Some information about the motion of the sediment is required for progress. For example, if the sediment sticks to the boundary $x_S(t)$ behaves like in a cylindrical centrifuge, and δA is x-independent for $x > x_P$; the particular shape of the tube becomes important.

Another possibility is that the sediment slips along the side walls to the outer wall, without remixing. Under the assumption that the sediment accumulates on the outer wall in a uniform layer of maximal volume fraction α_M behind the shock $x_S(t)$, we can estimate the position of this shock by the following dispersed phase volume balance

$$\alpha_M[R - x_S(t)] + \alpha(t)[x_S(t) - x_P(t)] = \alpha(0)(R - x_i) \tag{20}$$

which, after arrangement and use of (15), yields

$$x_S(t) = \left\{ R[\alpha_M - \alpha(0)] + \alpha(0) x_i [1 - (\alpha(t)/\alpha(0))^{1/2}] \right\} \frac{1}{\alpha_M - \alpha(t)}. \tag{21}$$

The value of $\alpha(t)$ is provided by (5).

The foregoing analysis may be formally applied in a similar manner to a suspension of "light" particles with the understanding that the particles settle now on the inner wall while pure fluid domains appear adjacent to the outer and side walls. However, this transformation is not trivial because thick pure fluid domains adjacent to the side wall may be unstable; instead, thin layers with a Boycott-effect type flow may develop. This is a complicated topic that needs careful theoretical and experimental investigation.

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